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LARGE THERMAL BUCKLING OF NONUNIFORM BEAMS AND PLATES

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Abstract—This article furnishes detailed global descriptions of the properties of buckled states of nonlinearly thermoelastic beams and plates when heated at their ends and edges. Only the axisymmetric deformation of circular plates is considered. In contrast to the models previously studied in the literature, those used here furnish a geometrically exact description of the deformation and allow a very general material response. The beams and plates may be nonuniform. The analysis relies on the combination of classical results for ordinary differential equations and with new results from bifurcation theory. The presentation emphasizes the crucial role of constitutive assumptions in the analysis. The development exhibits a number of novel features of physical importance not observed in more primitive models.

1. INTRODUCTION

In this paper we study the buckled states in the large of nonlinearly thermoelastic beams and circular plates due to elevations of temperature on the noninsulated parts of their boundaries. We are able to obtain quite detailed qualitative pictures of the buckled states, even when the deformations are exceedingly large and the bodies have nonuniform thickness, by combining classical results for ordinary differential equations with new results for global bifurcation theory. Since there have been several recent applications of global bifurcation theory to nonlinear elasticity (see [3, 5–7]), we restrict our attention almost exclusively to those aspects of the present problem that are novel. We emphasize the critical way the constitutive properties inform the entire analysis.

Our strategy is to characterize the global qualitative behavior of buckled states by the number of zeros of certain functions and examine how this number can change with the parameters. To fix ideas let k be a nonnegative integer. Then the function $\sin(k + 1)\pi s$ on the closed interval $[0, 1]$ vanishes at 0 and 1 and has k zeros on the open interval $(0, 1)$, each of these zeros being simple. Now any function $f(s)$ (with $f'(0) > 0$) having these same nodal properties looks like $\sin(k + 1)\pi s$. We may think of $\sin(k + 1)\pi s$ as the kth eigenmode describing the shape of a simple structure at the outset of a buckling process and we may think of $f(s)$ as describing the shape that evolves from $\sin(k + 1)\pi s$ as the deformation becomes large. There is no a priori reason to expect $f(s)$ to look like $\sin(k + 1)\pi s$ as the deformation develops. We show, however, that for the large buckling of beams and plates we can find a function f (in particular, a tangent angle) that preserves the qualitative (nodal) properties it inherits from the eigenfunctions for the linearized buckling problem, even when the deformation becomes very large. We do this for theories of beams and plates that are geometrically exact (in the sense that no contributions to the strains are discarded and no geometric expression such as $\sin\theta$ is replaced by its approximations θ or $\theta - \theta^3/6$). Moreover, we employ a very general class of nonlinearly thermoelastic constitutive equations.

Since the qualitative properties of solutions are preserved for all our constitutive equations, how do the constitutive equations affect our results? The constitutive equations determine how the deformation depends upon the thermal loads. In particular, we show in Section 11 that there is a threshold, expressed in terms of constitutive moduli, that separates subcritical from supercritical bifurcation. (Materials permitting subcritical bifurcation may have an effective buckling load well below that predicted by linear analysis.) Further aspects of the question are discussed in Section 12.

To see how nodal properties can be combined with bifurcation theory, we represent the governing equations abstractly as

$$\mathbf{u} = \mathbf{f}(\boldsymbol{\lambda}, \mathbf{u}) \tag{1.1}$$

where $\mathbf{u}$ represents the collection of basic unknowns, which we may regard specifying a configuration and a temperature field, and where $\boldsymbol{\lambda}$ represents the parameters of the problem, such as the boundary temperatures and the thickness variation of the beam or plate. $\mathbf{f}$ is a nonlinear operator. We adjust $\mathbf{u}$ and $\mathbf{f}$ so that $\mathbf{f}(\boldsymbol{\lambda}, \mathbf{0}) = \mathbf{0}$ for all $\boldsymbol{\lambda}$. Thus (1.1) admits the trivial solution $\mathbf{u} = \mathbf{0}$ for all values of the parameter $\boldsymbol{\lambda}$. These trivial solutions correspond to unbuckled states. If $(\boldsymbol{\lambda}, \mathbf{u})$ satisfies (1.1), then it is called a *solution pair*. We are interested in studying the properties of connected sets of solution pairs. We show that the amount of available qualitative information is almost as much as that for the linear eigenvalue problems that describe the onset of buckling.

Suppose that the original system of differential equations and boundary conditions has been converted to a "nice" set of integral equations and that (1.1) is this system. (Technically speaking, (1.1) is "nice" if $\mathbf{f}$ is completely continuous.) Global bifurcation theory [1, 2, 14] then says that connected families of solution pairs of (1.1) bifurcate from the trivial solutions at eigenvalues of odd algebraic multiplicity of the linearization of (1.1) and that these families do not stop abruptly, do not have holes, and have everywhere a dimension at least that of the number of components of $\boldsymbol{\lambda}$. If all parameters save one are fixed, then the connected families of nontrivial solutions have at least one of the following properties: (i) they are unbounded in the space of $\mathbf{u}$ and $\boldsymbol{\lambda}$; (ii) they connect the bifurcation point with another one corresponding to a different eigenvalue. Finally, if the eigenvalue at the bifurcation point is simple, then near the bifurcation point the solution $\mathbf{u}$ of (1.1) looks like the eigenfunction of the linearized problem.

We show that nodal properties of certain functions are preserved on connected sets of nontrivial solutions by reducing the question to a simple uniqueness theorem for ordinary differential equations. Thus the mathematical part of our study has three aspects: (i) a study of the linearized equations, which is easy, (ii) a study of nodal properties, which is likewise easy, (iii) a demonstration that the governing equations can be cast as nice integral equations. This last aspect, rather than being a technical mathematical exercise, in fact lies closest to the underlying physics. It devolves crucially upon the effective exploitation of physically reasonable constitutive assumptions. (Incidentally, in our treatment of plates it will be quite clear what a system (1.1) that is not nice would look like.) Only in the final steps of our treatment do we mention complete continuity for the sake of precision. The reader unfamiliar with the formal definition and its uses can safely ignore its brief appearance.

In Sections 2–5 we formulate in parallel the governing equations for beams and plates in order to exhibit their analogous structure. Our theories are geometrically exact and are based upon very general constitutive assumptions. In our set-up, we do not need to use the consequences of the Clausius–Duhem version, e.g. of the Second Law of Thermodynamics. In our study of plates we encounter some surprisingly delicate questions of stability devolving on both constitutive assumptions and the nonuniformity of the plate. On the other hand, the role of heat conduction is more pronounced in the beam problems we study because we allow for nonsymmetric boundary conditions in these problems.

We use the convention that equations identified by a "b" and "p" are respectively valid only for beams and plates. C^0 denotes the collection of continuous functions on $[0, 1]$.

2. GEOMETRY OF DEFORMATION

Let $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ be a fixed, right-handed, orthonormal basis for Euclidean 3-space. We study the buckling in the $(\mathbf{i}, \mathbf{j})$-plane of nonlinearly thermoelastic beams that can suffer flexure, extension, and shear. A configuration of such a beam is specified by a position vector function $\mathbf{r}$ and a unit vector function $\mathbf{b}$ of the real variable s in $[0, 1)$ with $\mathbf{r}\cdot\mathbf{k}$

$= 0 = \mathbf{b}\cdot\mathbf{k}$. s is interpreted as a scaled arc length parameter of the straight line of centroids of a beam in its natural reference configuration. Thus s identifies material sections of the beam. The vector $\mathbf{r}(s)$ is interpreted as the position in the deformed configuration of the material point at the centroid of the section s. The vector $\mathbf{b}(s)$, which with $\mathbf{k}$ determines a plane, is interpreted as characterizing the deformed configuration of the section s.

A circular plate, with axis of symmetry $\mathbf{j}$, that undergoes an axisymmetric deformation has each configuration determined by that of the intersection of the plate with any half-plane having $\mathbf{j}$ as an edge. We take as a reference the half-plane containing $\mathbf{i}$. Then the deformation of this section of the plate can be given exactly the same description as that of the beam in terms of $\mathbf{r}$ and $\mathbf{b}$. The reference configurations of both the beam and plate are given by

$$\mathbf{r}(s) = s\mathbf{i}, \qquad \mathbf{b}(s) = \mathbf{j}. \tag{2.1}$$

We introduce the unit vector $\mathbf{a} = \mathbf{b} \times \mathbf{k}$ (normal to sections) and the angle ψ it makes with $\mathbf{i}$ by

$$\mathbf{a}(s) = \cos\psi(s)\mathbf{i} + \sin\psi(s)\mathbf{j}, \qquad \mathbf{b}(s) = -\sin\psi(s)\mathbf{i} + \cos\psi(s)\mathbf{j}. \tag{2.2}$$

We denote differentiation with respect to s by a prime. We decompose the vector $\mathbf{r}'(s)$ tangent to $\mathbf{r}$ at s as

$$\mathbf{r}'(s) = \nu(s)\mathbf{a}(s) + \eta(s)\mathbf{b}(s). \tag{2.3}$$

η measures shear; $\nu = \mathbf{r}' \cdot \mathbf{a}$ really measures a volume change, but can be thought of as measuring elongation. (These variables prove to be more natural than the axial elongation and the shear angle.) The requirements that the local ratio of deformed to reference length of $\mathbf{r}$ at s be positive and that the section at s not be sheared so severely that $\mathbf{b}(s)$ is parallel to $\mathbf{r}'(s)$ is ensured by

$$\nu(s) > 0. \tag{2.4}$$

The variables ν, η,

$$\mu \equiv \psi' \tag{2.5}$$

are the *strains* for our problem for beams. For plates we add the circumferential stretch

$$\tau(s) \equiv \mathbf{r}(s) \cdot \mathbf{i}/s \tag{2.6p}$$

and a measure of bending about rays

$$\sigma(s) \equiv \sin\psi(s)/s. \tag{2.7p}$$

We complement (2.4) with

$$\tau(s) > 0. \tag{2.8p}$$

More sophisticated versions of (2.4) and (2.8p) are available. See [3, 10]. For both the beam and plate we take boundary conditions

$$\mathbf{r}(0) = \mathbf{0}, \qquad \mathbf{r}(1) \cdot \mathbf{i} = l, \tag{2.9}$$

$$\psi(0) = 0, \qquad \psi(1) = 0. \tag{2.10}$$

3. EQUILIBRIUM EQUATIONS

For a beam undergoing a planar deformation we let $\mathbf{n}(s)$ with $\mathbf{n} \cdot \mathbf{k} = 0$ denote the resultant contact force and $M\mathbf{k}$ denote the resultant contact couple exerted on the material of $[0, s]$ by that of $(s, 1]$. (We are assuming that the material properties of the beam have enough symmetry that resultants of this form correspond to the planar deformations described in Section 2.)

To describe the axisymmetric deformation of a circular plate, we suppose that the axis of symmetry of the plate is $\mathbf{j}$. We introduce the polar coordinate ϕ and the base vectors $\mathbf{e}_1(\phi) = \cos\phi\, \mathbf{i} + \sin\phi(-\mathbf{k})$, $\mathbf{e}_2(\phi) = -\sin\phi\, \mathbf{i} + \cos\phi(-\mathbf{k})$. We let $\mathbf{n}(s)$, with $\mathbf{n}(s) \cdot \mathbf{k} = 0$, denote the resultant contact force and $M(s)\mathbf{k}$ denote the resultant contact couple per unit reference length of the circle of radius s that is exerted at $s\mathbf{i}$ on the material inside this circle by that outside the circle. Let $T(s)\mathbf{e}_2(\phi_0)$, denote the resultant contact force and $\mathbf{m}_2(s, \phi_0)$, with $\mathbf{m}_2(s, \phi_0) \cdot \mathbf{e}_2(\phi_0) = 0$, denote the resultant contact couple per unit reference length of the ray $\phi = \phi_0$ that is exerted at $s\mathbf{e}_1(\phi_0)$ on the material with $\phi \leq \phi_0$ by that with $\phi > \phi_0$. Set $\Sigma(s) \cos\psi(s) = \mathbf{e}_1 \cdot \mathbf{m}_2$. (We are assuming that the material properties of the plate yield resultants of these forms when the plate undergoes an axisymmetric deformation. The interpretation of Σ given here corrects a faulty one given in [3].)

We now set

$$\mathbf{n}(s) = N(s)\mathbf{a}(s) + H(s)\mathbf{b}(s). \tag{3.1}$$

Let us assume that the only forces applied to the beam are applied at $s = 0, 1$ and that the only forces applied to the plate are applied at $s = 1$. Then the classical forms of the equilibrium of forces for the beam and plate are respectively

$$\mathbf{n}' = \mathbf{0}, \tag{3.2b}$$

$$(s\mathbf{n})' - T\mathbf{i} = \mathbf{0}. \tag{3.2p}$$

(After obtaining the equilibrium equations for plates, we replace $\mathbf{e}_1$ by $\mathbf{i}$ and $\mathbf{e}_2$ by $-\mathbf{k}$.)

We assume that there is no shear force preventing the vertical motion of the material of $s = 1$:

$$H(1) = 0. \tag{3.3}$$

Then (2.10) implies that

$$\mathbf{n}(1) = N(1)\mathbf{i}. \tag{3.4}$$

We then get from (3.2) the integral equations of equilibrium

$$\mathbf{n}(s) = N(1)\mathbf{i}, \tag{3.5b}$$

$$s\mathbf{n}(s) = \left[N(1) - \int_s^1 T(t)\,dt\right]\mathbf{i}. \tag{3.5p}$$

The corresponding componential versions of these equations are

$$N(s) = N(1)\cos\psi(s), \tag{3.6b}$$

$$sN(s) = \left[N(1) - \int_s^1 T(t)\,dt\right]\cos\psi(s), \tag{3.6p}$$

$$H(s) = -N(1)\sin\psi(s), \tag{3.7b}$$

$$sH(s) = -\left[N(1) - \int_s^1 T(t)\,dt\right]\sin\psi(s). \tag{3.7p}$$

If there are no external couples applied to the beam and plate over (0, 1), then the classical forms of the equilibrium of torques for the beam and plate are

$$M' + \mathbf{k} \cdot (\mathbf{r}' \times \mathbf{n}) = 0, \tag{3.8b}$$

$$(sM)' - \Sigma \cos \psi + s\mathbf{k} \cdot (\mathbf{r}' \times \mathbf{n}) = 0. \tag{3.8p}$$

The use of (2.3) and (3.5) reduces these equations to

$$M' - N(1)[\nu \sin \psi + \eta \cos \psi] = 0, \tag{3.9b}$$

$$(sM)' - \Sigma \cos \psi - \left[N(1) - \int_s^1 T(t)\, dt\right] [\nu \sin \psi + \eta \cos \psi] = 0. \tag{3.9p}$$

Note that by setting $s = 0$ in (3.6p) we obtain the identity

$$N(1) - \int_s^1 T(t)\, dt = \int_0^s T(t)\, dt, \tag{3.10p}$$

which can be used to simplify (3.6p), (3.7p), (3.9p) by eliminating $N(1)$ from these equations.

For a more detailed derivation of these equations, see [3, 10].

4. HEAT CONDUCTION

Let $\theta(s)$ be the temperature at s (for either the beam or plate). θ is required to be positive. We set

$$\gamma(s) = \theta'(s). \tag{4.1}$$

Let $\Gamma(s)$ represent the rate at which heat crosses the section s from $[s, 1]$ to $[0, s)$ for the beam and the corresponding rate per unit reference length of the circle of radius s for the plate. We assume that the lateral surface of the beam and the faces of the plate are thermally insulated and that there are no heat sources. For the plate the assumption of axisymmetry precludes the possibility of heat flow in the circumferential direction. Then the energy equations are

$$\Gamma' = 0, \tag{4.2b}$$

$$(s\Gamma)' = 0. \tag{4.2p}$$

From (4.2b) we immediately obtain

$$\Gamma(s) = \Gamma(1). \tag{4.3b}$$

We shall seek bounded Γ's for the plate, in which case (4.2p) implies that

$$\Gamma(s) = 0. \tag{4.3p}$$

For the beam we prescribe temperatures at each end:

$$\theta(0) = \alpha - \beta, \qquad \theta(1) = \alpha. \tag{4.4b}$$

Without loss of generality we assume that $\beta \geq 0$, $\alpha - \beta > 0$. For the plate we merely prescribe the temperature at the outer edge:

$$\theta(1) = \alpha. \tag{4.4p}$$

Were we to study annular plates, then (4.3p) need not hold and we would have a pair of boundary conditions like (4.4b). The treatment of such problems offers no novelties. We shall see that the complete plate presents us with some fascinating singularities at the origin, which have both physical and mathematical implications.

5. CONSTITUTIVE EQUATIONS

For the beam theory we assume that there are constitutive functions $\hat{N}$, $\hat{H}$, $\hat{M}$, $\hat{\Gamma}$ of ν, η, μ, γ, θ, s such that

$$N(s) = \hat{N}(\nu(s), \eta(s), \mu(s), \gamma(s), \theta(s), s), \text{ etc.} \tag{5.1b}$$

The domain of $\hat{N}$, $\hat{H}$, $\hat{M}$, $\hat{\Gamma}$ consists of all ν, η, μ, γ, θ, s such that $\nu > 0$, $\theta > 0$, $s \in [0, 1]$. For simplicity we assume that these functions are thrice continuously differentiable. For the plate theory we assume that there are constitutive functions $\hat{N}$, $\hat{H}$, $\hat{M}$, $\hat{T}$, $\hat{\Sigma}$, $\hat{\Gamma}$ of ν, η, μ, τ, σ, γ, θ, s such that

$$N(s) = \hat{N}(\nu(s), \eta(s), \mu(s), \tau(s), \sigma(s), \gamma(s), \theta(s), s), \text{ etc.} \tag{5.1p}$$

The domain of these functions consists of all ν, η, μ, τ, σ, γ, θ, s such that $\nu > 0$, $\tau > 0$, $\theta > 0$, $s \in [0, 1]$. These functions are also taken to be thrice continuously differentiable.

The explicit dependence of these constitutive functions on s allows the material to be nonuniform. In rod and shell theories the most natural source of such nonuniformity is variable thickness. The Clausius–Duhem version of the Second Law of Thermodynamics, as well as other versions, would imply that only $\hat{\Gamma}$ depends on γ and that the stress resultants $\hat{N}$, $\hat{H}$, $\hat{M}$, $(\hat{T}, \hat{\Sigma})$ can be represented as the derivatives of the free energy function with respect to ν, η, μ, (τ, σ), respectively. Since we need neither of these conclusions in our analysis we do not commit ourselves to any version of the Second Law.

We require that the constitutive functions satisfy the following monotonicity condition: (The symmetric part of)

$$\frac{\partial(\hat{N}, \hat{H}, \hat{M}, \hat{\Gamma})}{\partial(\nu, \eta, \mu, \gamma)} \text{ is positive-definite.} \tag{5.2}$$

The first term of (5.2) represents the *matrix* of partial derivatives of $\hat{N}$, $\hat{H}$, $\hat{M}$, $\hat{\Gamma}$ with respect to ν, η, μ, γ. Condition (5.2) implies that $\hat{N}$ is an increasing function of ν (when its other arguments are fixed), . . . , $\hat{\Gamma}$ is an increasing function of γ. Moreover, the effect on $\hat{N}$ due to a change in ν is more pronounced than that due to a change in any of the arguments η, μ, γ. Analogous remarks apply to the other constitutive functions. In particular, the dependence of $\hat{N}$, $\hat{H}$, $\hat{M}$ on γ is relatively weak. Thus (5.2) embodies a generalization of the requirement that these variables be independent of γ. If the beam or plate theory is obtained from a general three-dimensional theory of thermoelasticity, then (5.2) is a direct consequence of the Strong Ellipticity Condition (see [3, 10]). For plates the Strong Ellipticity Condition also implies that

$$\frac{\partial(\hat{T}, \hat{\Sigma})}{\partial(\tau, \sigma)} \text{ is positive-definite.} \tag{5.3p}$$

We shall not need (5.3p) in our work. In our analysis of plates we discuss the implications of other restrictions, which we do not consider to be universally applicable.

For the beam theory we further require that extremes of strains and temperature gradients be accompanied by extremes of their corresponding constitutive functions:

$$\hat{N} \to \begin{Bmatrix} \infty \text{ uniformly as } \nu \to \infty \\ -\infty \text{ uniformly as } \nu \to 0 \end{Bmatrix},$$

$$\hat{H}, \hat{M}, \hat{\Gamma} \to \pm\infty \text{ uniformly as} \tag{5.5b}$$

$$\eta, \mu, \gamma \to \pm\infty, \text{ respectively.}$$

Conditions (5.2), (5.4), (5.5) justify a global implicit function theorem to the effect that if N, H, M, Γ, s are prescribed, then the algebraic equations

$$\hat{N}(\nu, \eta, \mu, \gamma, \theta, s) = N, \hat{H}(\nu, \eta, \mu, \gamma, \theta, s) = H, \hat{M}(\nu, \eta, \mu, \gamma, \theta, s) = M, \quad \hat{\Gamma}(\nu, \eta, \mu, \gamma, \theta, s) = \Gamma \tag{5.6b}$$

have a unique solution (ν, η, μ, γ), which we denote by

$$\nu = \hat{\nu}(N, H, M, \Gamma, \theta, s), \quad \eta = \hat{\eta}(N, H, M, \Gamma, \theta, s), \quad \mu = \hat{\mu}(N, H, M, \Gamma, \theta, s), \quad \gamma = \hat{\gamma}(N, H, M, \Gamma, \theta, s). \tag{5.7b}$$

Moreover, $\hat{\nu}$, $\hat{\eta}$, $\hat{\mu}$, $\hat{\gamma}$ are continuously differentiable. (See [13] e.g.). These results can be generalized to the constitutive equations for plates, but we do not do so because our analysis for plates differs from that for beams and we do not require the analog of (5.7b).

In consonance with (5.4) and (5.5) we assume that

$$\hat{\nu} \to \begin{Bmatrix} \infty \text{ uniformly as } N \to \infty \\ 0 \text{ uniformly as } N \to -\infty \end{Bmatrix}, \tag{5.8b}$$

$$\hat{\eta}, \hat{\mu}, \hat{\gamma} \to \pm\infty \text{ uniformly as } H, M, \Gamma \to \pm\infty, \text{ respectively.} \tag{5.9b}$$

For beams we assume that

$$\eta(N, 0, 0, \Gamma, \theta, s) = 0, \mu(N, 0, 0, \Gamma, \theta, s) = 0, \gamma(N, 0, 0, 0, \theta, s) = 0. \tag{5.10b}$$

(It is reasonable to make the far more restrictive assumptions that $\hat{\eta}(N, 0, M, \Gamma, \theta, s) = 0$, $\hat{\mu}(N, H, 0, \Gamma, \theta, s) = 0$, $\hat{\gamma}(N, H, M, 0, \theta, s) = 0$, in which case (5.2) implies that the shear strain η vanishes if and only if the shear force H vanishes, etc.). For plates we make the related assumption that $\hat{H}$, $\hat{M}$, $\hat{\Sigma}$, $\hat{\Gamma}$ are such that

$$(H, M, \Sigma, \Gamma) = (0, 0, 0, 0) \text{ when } (\eta, \mu, \sigma, \gamma) = (0, 0, 0, 0). \tag{5.10p}$$

A plate constrained to undergo only axisymmetric deformations is *isotropic at its center* if

$$\begin{aligned} \hat{N}(\nu, 0, \mu, \tau, \sigma, 0, \theta, 0) &= \hat{T}(\tau, 0, \sigma, \nu, \mu, 0, \theta, 0), \\ \hat{N}_s(\nu, 0, \mu, \tau, \sigma, 0, \theta, 0) &= \hat{T}_s(\tau, 0, \sigma, \nu, \mu, 0, \theta, 0), \\ \hat{M}(\nu, 0, \mu, \tau, \sigma, 0, \theta, 0) &= \hat{\Sigma}(\tau, 0, \sigma, \nu, \mu, 0, \theta, 0), \\ \hat{M}_s(\nu, 0, \mu, \tau, \sigma, 0, \theta, 0) &= \hat{\Sigma}_s(\tau, 0, \sigma, \nu, \mu, 0, \theta, 0). \end{aligned} \tag{5.11p}$$

Here and below subscripts denote partial derivatives.

6. INTEGRAL EQUATIONS FOR BEAMS

The boundary value problem for beams consists of the geometric relations (2.3), (2.5), the geometric boundary conditions (2.9), (2.10), the equilibrium equations (3.6b), (3.7b), (3.9b), the energy equation (4.3b), the temperature boundary conditions (4.4b), and the constitutive equations (5.1b) or (5.7b). We now recast this problem into a form suitable for the application of the theory described in Section 1.

Let us set

$$M(s) = M(1) + \Delta(s). \tag{6.1}$$

We begin our analysis of the governing equations by determining the unknown constants $N(1)$, $M(1)$, $\Gamma(1)$ as functionals of ψ, Δ, θ. From (2.9), (2.3), (5.7), (3.6b), (3.7b), (4.3b) we have

$$\begin{aligned} l &= \int_0^1 \mathbf{r}'(s)\cdot \mathbf{i}\, ds = \int_0^1 [\nu(s)\cos\psi(s) - \eta(s)\sin\psi(s)]\, ds \\ &= \int_0^1 [\hat{\nu}(N(1)\cos\psi(s), -N(1)\sin\psi(s), M(1)+\Delta(s), \\ &\quad \Gamma(1), \theta(s), s)\cos\psi(s) - \hat{\eta}(N(1)\cos\psi(s), -N(1)\sin\psi(s), \\ &\quad M(1)+\Delta(s), \Gamma(1), \theta(s), s)\sin\psi(s)]\, ds \\ &\equiv X(N(1), M(1), \Gamma(1); \psi, \Delta, \theta). \end{aligned} \tag{6.2}$$

Similarly from (2.10), (2.5), (5.7), we obtain

$$\begin{aligned} 0 &= \int_0^1 \psi'(s)\, ds = \int_0^1 \hat{\mu}(N(1)\cos\psi(s), -N(1)\sin\psi(s), \\ &\quad M(1)+\Delta(s), \Gamma(1), \theta(s), s)\, ds \\ &\equiv \Psi(N(1), M(1), \Gamma(1); \psi, \Delta, \theta) \end{aligned} \tag{6.3}$$

and from (4.4b), (4.1), (5.7) we obtain

$$\begin{aligned} \beta &= \int_0^1 \hat{\gamma}(N(1)\cos\psi(s), -N(1)\sin\psi(s), M(1)+\Delta(s), \Gamma(1), \theta(s), s)\, ds \\ &\equiv \Theta(N(1), M(1), \Gamma(1); \psi, \Delta, \theta). \end{aligned} \tag{6.4}$$

Let A, B, C be real numbers. Then the chain rule shows that the quadratic form

$$\begin{aligned} & A^2 \frac{\partial X}{\partial N(1)} + AB\frac{\partial X}{\partial M(1)} + AC\frac{\partial X}{\partial \Gamma(1)} + BA\frac{\partial \Psi}{\partial N(1)} + B^2\frac{\partial \Psi}{\partial M(1)} \\ &\quad + BC\frac{\partial \Psi}{\partial \Gamma(1)} + CA\frac{\partial \Theta}{\partial N(1)} + CB\frac{\partial \Theta}{\partial M(1)} + C^2\frac{\partial \Theta}{\partial \Gamma(1)} \\ &= \int_0^1 \left\{ A\cos\psi \left[A\cos\psi\frac{\partial\hat{\nu}}{\partial N} - A\sin\psi\frac{\partial\hat{\nu}}{\partial H} + B\frac{\partial\hat{\nu}}{\partial M} + C\frac{\partial\hat{\nu}}{\partial\Gamma}\right]\right. \\ &\quad - A\sin\psi\left[A\cos\psi\frac{\partial\hat{\eta}}{\partial N} - A\sin\psi\frac{\partial\hat{\eta}}{\partial H} + B\frac{\partial\hat{\eta}}{\partial M} + C\frac{\partial\hat{\eta}}{\partial\Gamma}\right] \\ &\quad + B\left[A\cos\psi\frac{\partial\hat{\mu}}{\partial N} - A\sin\psi\frac{\partial\hat{\mu}}{\partial H} + B\frac{\partial\hat{\mu}}{\partial M} + C\frac{\partial\hat{\mu}}{\partial\Gamma}\right] \\ &\quad + C\left[A\cos\psi\frac{\partial\hat{\gamma}}{\partial N} - A\sin\psi\frac{\partial\hat{\gamma}}{\partial H} + B\frac{\partial\hat{\gamma}}{\partial M} + C\frac{\partial\hat{\gamma}}{\partial\Gamma}\right] ds. \end{aligned} \tag{6.5}$$

Now (5.2) implies that $\partial(\hat{\nu}, \hat{\eta}, \hat{\mu}, \hat{\gamma})/\partial(N, H, M, \Gamma)$ is positive-definite. Hence the right side of (6.5) is positive for $(A, B, C) \neq (0, 0, 0)$. Thus

$$\partial(X, \Psi, \Theta)/\partial(N(1), M(1), \Gamma(1)) \text{ is positive-definite.} \tag{6.6}$$

Moreover, (5.5) implies that

$$\begin{aligned} &\hat{\nu}(N(1)\cos\psi(s), -N(1)\sin\psi(s), M(1)+\Delta(s), \Gamma(1), \theta(s), s)\cos\psi(s) \\ &\to \begin{cases} 0 \text{ uniformly as } N(1)\to -\infty & \text{if } \cos\psi(s) \geqslant 0, \\ -\infty \text{ uniformly as } N(1)\to -\infty & \text{if } \cos\psi(s) < 0, \\ \infty \text{ uniformly as } N(1)\to \infty & \text{if } \cos\psi(x) > 0, \\ 0 \text{ uniformly as } N(1)\to \infty & \text{if } \cos\psi(s) \leqslant 0; \end{cases} \end{aligned} \tag{6.7}$$

$$-\hat{\eta}(N(1)\cos\psi(s),\ -N(1)\sin\psi(s),\ M(1)+\Delta(s),\ \Gamma(1),\ \theta(s),\ s)\sin\psi(s)$$
$$\to \pm\infty \text{ uniformly as } N(1)\to\pm\infty \qquad \text{if } \sin\psi(s)\neq 0; \tag{6.8}$$
$$\hat{\mu}(N(1)\cos\psi(s),\ -N(1)\sin\psi(s),\ M(1)+\Delta(s),\ \Gamma(1),\ \theta(s),\ s)$$
$$\to \pm\infty \text{ uniformly as } M(1)\to\pm\infty, \tag{6.9}$$
$$\hat{\gamma}(N(1)\cos\psi(s),\ -N(1)\sin\psi(s),\ M(1)+\Delta(s),\ \Gamma(1),\ \theta(s),\ s)$$
$$\to \pm\infty \text{ uniformly as } \Gamma(1)\to\pm\infty. \tag{6.10}$$

Thus (6.2)–(6.4) imply that if ψ and M are continuous, then

$$X(N(1), M(1), \Gamma(1); \psi, \Delta, \theta)\to\pm\infty \text{ uniformly}$$
$$\text{as } N(1)\to\pm\infty \quad \text{if } \psi\neq 0, \tag{6.11}$$
$$X(N(1), M(1), \Gamma(1); \psi, \Delta, \theta)\to\begin{Bmatrix}\infty\\ 0\end{Bmatrix} \text{ uniformly}$$
$$\text{as } N(1)\to\begin{Bmatrix}\infty\\ -\infty\end{Bmatrix} \quad \text{if } \psi = 0, \tag{6.12}$$
$$\Psi(N(1), M(1), \Gamma(1); \psi, \Delta, \theta)\to\pm\infty \text{ uniformly as } M(1)\to\pm\infty, \tag{6.13}$$
$$\Theta(N(1), M(1), \Gamma(1); \psi, \Delta, \theta)\to\pm\infty \text{ uniformly as } \Gamma(1)\to\pm\infty. \tag{6.14}$$

Conditions (6.6), (6.11)–(6.14) justify a global implicit function theorem to the effect that (6.2)–(6.4) have a unique solution, which we denote as

$$\begin{aligned} N(1) &= N_1[\psi, \Delta, \theta; l, \beta], \quad M(1) = M_1[\psi, \Delta, \theta; l, \beta], \\ \Gamma(1) &= \Gamma_1[\psi, \Delta, \theta; l, \beta] \end{aligned} \tag{6.15}$$

with N_1, M_1, Γ_1 being continuously differentiable functionals on their domain $C^0\times C^0\times C^0\times(0,\infty)\times[0,\infty]$. We henceforth suppress the arguments l and β whenever their presence is not essential.

We can now reduce the governing equations, listed in the first paragraph of this section, to the integral equations:

$$\psi(s) = \int_0^s \hat{\mu}(N_1[\psi, \Delta, \theta]\cos\psi(t),\ -N_1[\psi, \Delta, \theta]\sin\psi(t),$$
$$M_1\mathrm{p}[\psi, \Delta, \theta]+\Delta(t),\ \Gamma_1[\psi, \Delta, \theta],\ \theta(t),\ t)\,dt, \tag{6.16}$$
$$\Delta(s) = -N_1[\psi, \Delta, \theta]\int_s^1[\hat{\nu}\sin\psi(t)+\hat{\eta}\cos\psi(t)]\,dt, \tag{6.17}$$
$$\theta(s) = \alpha-\beta+\int_0^s\hat{\gamma}\,dt. \tag{6.18}$$

The arguments of $\hat{\nu}$ and $\hat{\eta}$ in (6.17) and of $\hat{\gamma}$ in (6.18) are the same as those of $\hat{\mu}$ in (6.16). (6.16) comes from (2.5) and (5.7), (6.17) comes from (3.9b), and (6.18) comes from (4.4b) and (5.7).

7. TRIVIAL SOLUTIONS OF THE BEAM EQUATIONS

A solution of (6.16)–(6.18) is called *trivial* if

$$\psi = 0 = \Delta. \tag{7.1}$$

Such a solution describes an unbuckled state. Let us substitute (7.1) into (6.3) to get

$$\Psi(N(1), M(1), \Gamma(1); 0, 0, \theta) = 0. \tag{7.2}$$

But the second condition of (5.10b) ensures that $\Psi(N(1), 0, \Gamma(1); 0, 0, \theta) = 0$ while the positivity of $\partial\Psi/\partial M(1)$, following from (6.6), ensures that (7.2) is equivalent to

$$M_1[0, 0, \theta; l, \beta] = 0. \tag{7.3}$$

It now follows from (5.10b) that if (7.1) holds, then (6.16) and (6.17) are identically satisfied and that (6.18) reduces to

$$\theta(s) = \alpha - \beta + \int_0^s \hat{\gamma}(N_1[0, 0, \theta; l, \beta], 0, 0, \Gamma_1[0, 0, \theta; l, \beta], \theta(t), t)\, dt. \tag{7.4}$$

Note that if $\hat{\gamma}$ is independent of s and if $\beta \neq 0$, then (7.4) admits the solution $\theta(s) = \alpha - \beta(1 - s)$ if and only if $\hat{\gamma}$ is also independent of θ.

The boundary conditions that $\mathbf{r}(1) \cdot \mathbf{i} = l$ and $\theta(1) = \alpha$ are accounted for in N_1 and Γ_1; see (6.2)–(6.4), (6.15).

We seek solutions θ of (7.4) in C^0. Now (5.2) implies that $\partial\hat{\gamma}/\partial\Gamma > 0$. Thus (5.10b) implies that $\hat{\gamma}(N(1), 0, 0, \Gamma(1), \theta(s), s)$ has the same sign as $\Gamma(1)$. It then follows from the definition of $\Gamma_1[0, 0, \theta; l, \beta]$ that the right side of (7.4) lies in $[\alpha - \beta, \alpha]$ for each s and for any θ in C^0. It is then easy to see that the right side of (7.4) defines a continuous and compact mapping taking the set of continuous θ's satisfying $\alpha - \beta \leq \theta \leq \alpha$ into itself. By the Schauder Fixed Point Theorem (7.4) has a solution θ_0, which is actually twice continuously differentiable. Further constitutive restrictions are needed to show that it is unique.

We may think of the trivial problem as being parametrized by l, α, β and the kind of nonuniformity the beam possesses. This nonuniformity is expressed by the explicit dependence of the constitutive functions on s. More generally, we may regard the problem as having the constitutive functions themselves as parameters. A theory much like that discussed in Section 1 (which is described fully in the paragraph containing (8.27)) enables us to conclude that for each value of our infinite-dimensional parameter there is a trivial solution and that the corresponding set of solution pairs is "maximally" connected. We shall see that the situation for plates is not so favorable.

8. INTEGRAL EQUATIONS FOR PLATES, TRIVIAL STATES

The boundary value problem for plates consists of the geometric relations (2.3), (2.5)–(2.7), the geometric boundary conditions (2.9), (2.10), the equilibrium equations (3.6p), (3.7p), (3.9p), the energy equation (4.3p), the temperature boundary condition (4.4p) and the constitutive equations (5.1p). In converting these equations into integral equations we do not follow the plan of Section 6 because we confront serious difficulties with the singularity at $s = 0$, which are exacerbated by the coupling between the strains ν, η, μ and τ, σ.

The basic difficulties and the methods for overcoming them are illustrated in microcosm by the equations for trivial solutions

$$\psi = 0, \qquad \theta = \alpha. \tag{8.1}$$

It then follows from (2.5), (2.7p), (3.7p), (4.1), (4.3p) that $\mu, \sigma, H, \gamma, \Gamma = 0$. Since (5.2) implies that $\hat{H}_\eta > 0$, we conclude from (5.10p) that the equation $\hat{H}(\nu, \eta, 0, \tau, 0, 0, \alpha, s) = 0$ implies that $\eta = 0$. Thus from (8.1) and (5.10p) we obtain

$$H, \eta, M, \mu, \Sigma, \sigma, \Gamma, \gamma = 0. \tag{8.2}$$

The only equations not satisfied identically are (2.6p), (3.6p), (see 3.10p, 5.1p), which we write as

$$sN(s) = \int_0^s T(t)\, dt\,, \tag{8.3}$$

$$(s\tau)' = \nu, \qquad \tau(1) = l. \tag{8.4}$$

$$N(s) = \hat{N}(\nu(s), \tau(s), \alpha, s), \qquad T(s) = \hat{T}(\nu(s), \tau(s), \alpha, s). \tag{8.5}$$

(In (8.5) and in the rest of this section we suppress the arguments η, μ, σ, γ, which vanish for trivial solutions.)

We rewrite these equations as

$$\{s\hat{N}([s\tau(s)]', \tau(s), \alpha, s)\}' = \hat{T}([s\tau(s)]', \tau(s), \alpha, s), \tag{8.6}$$

or equivalently,

$$(\hat{N}_\nu)s^{-2}(s^3\tau')' = (\hat{N}_\nu - \hat{N}_\tau)\tau' + s^{-1}(\hat{T} - \hat{N}) - \hat{N}_s \tag{8.7}$$

where the arguments of $\hat{T}$, $\hat{N}$, and their derivatives are those indicated in (8.6). Here and below, the subscripts on constitutive functions represent partial derivatives. The reciprocal of s appearing on the right side of (8.7) is a source of concern.

Now if the plate is uniform and isotropic, i.e. if $\hat{N}$ and $\hat{T}$ are independent of s, and if

$$\hat{N}(\nu, \tau, \theta) = \hat{T}(\tau, \nu, \theta), \tag{8.8}$$

then (8.3)–(8.5) admit the trivial solution

$$\nu = l = \tau. \tag{8.9}$$

Further assumptions are needed to show that (8.9) is unique. These are discussed below.

If the plate is not uniform or not isotropic, one can still show that there is a classical trivial solution for all l, α and for all materials meeting mild growth conditions by specializing the methods of [4] (which are based upon the theory of variational inequalities).

If we were to require further that

$$\frac{\partial(\hat{N}, \hat{T})}{\partial(\nu, \tau)} \text{ is positive-definite,} \tag{8.10}$$

then it is easy to show that (8.3)–(8.5) has a unique solution for each fixed $l > 0$, $\alpha > 0$ (see [4], Section 6). (Condition (8.10) is *not* a consequence of the Strong Ellipticity Condition.)

The basic existence result we have just quoted is misleadingly simple. It says nothing about the connectivity of the collection of *solution pairs* $((l, \alpha), \tau)$. The absence of good connectivity results suggests the possibility of instability. We accordingly investigate this question by giving an alternative formulation to (8.3)–(8.5). The methods we develop are basic to our goal of obtaining detailed global information about buckled states.

In view of (8.7) we set

$$u(s) = s^{-2}[s^3\tau'(s)]'. \tag{8.11}$$

Thus if $\lim_{s\to 0} s^3\tau'(s) = 0$, then

$$s^3\tau'(s) = \int_0^s t^2u(t)\,dt. \tag{8.12}$$

Integration by parts yields two alternative expressions for τ:

$$\tau(s) - \tau(0) = \frac{1}{2}\int_0^s (1 - s^{-2}t^2)u(t)\,dt, \tag{8.13}$$

$$\tau(s) = l - \frac{1}{2}\int_0^s (s^{-2} - 1)t^2u(t)\,dt - \frac{1}{2}\int_s^1 (1 - t^2)u(t)\,dt. \tag{8.14}$$

We replace τ and τ' wherever they appear in (8.7) with their representations from (8.12) to (8.14). We accordingly have a nonlinear integral equation for u, which we represent as

$$u = f((l, \alpha), u). \tag{8.15}$$

We seek solutions u in C^0. In this case (8.4), (8.12), (8.13) imply that

$$\nu(s) = s\tau'(s) + \tau(s) \to \tau(0) \text{ as } s \to 0. \tag{8.16}$$

We must show that the right side of (8.7) (or (8.15)) is well behaved at $s = 0$. For this purpose we use the isotropy conditions (5.11p), which imply that

$$\begin{aligned} \hat{T}(\nu, \tau, \theta, 0) &= \hat{N}(\tau, \nu, \theta, 0), \quad \hat{T}_s(\nu, \tau, \theta, 0) = \hat{N}_s(\tau, \nu, \theta, 0), \\ \hat{T}_\nu(\nu, \tau, \theta, 0) &= \hat{N}_\tau(\tau, \nu, \theta, 0), \quad \hat{T}_\tau(\nu, \tau, \theta, 0) = \hat{N}_\nu(\tau, \nu, \theta, 0). \end{aligned} \tag{8.17}$$

By Taylor's Theorem and (8.17) we have

$$\begin{aligned} \hat{T}(\nu(s), \tau(s), \alpha, s) = T^0 &+ T^0_\nu[\nu(s) - \nu(0)] + T^0_\tau[\tau(s) - \tau(0)] \\ &+ T^0_s s + \frac{1}{2}\overline{T}^{\nu\nu}(s)[\nu(s) - \nu(0)]^2 \\ &+ \overline{T}^{\nu\tau}(s)[\nu(s) - \nu(0)][\tau(s) - \tau(0)] \\ &+ \overline{T}^{\nu s}(s)\,[\nu(s) - \nu(0)]\,s + \frac{1}{2}\overline{T}^{\tau\tau}(s)[\tau(s) - \tau(0)]^2 \\ &+ \overline{T}^{\tau s}(s)[\tau(s) - \tau(0)]s + \frac{1}{2}\overline{T}^{ss}(s)s^2, \text{ etc.} \end{aligned} \tag{8.18}$$

where

$$T^0 \equiv \hat{T}(\tau(0), \tau(0), \alpha, 0), \text{ etc.} \tag{8.19}$$

$$\overline{T}^{\nu\nu}(s) \equiv \int_0^1 \hat{T}_{\nu\nu}(t\nu(s) + (1 - t)\tau(0), t\tau(s) + (1 - t)\tau(0), \alpha, ts)\, dt, \text{ etc.} \tag{8.20}$$

We now replace $\hat{T} - \hat{N}$ on the right side of (8.7) with its expansion about $(\tau(0), \tau(0), \alpha, 0)$ of the form (8.18) and replace $(\hat{N}_\nu - \hat{N}_\tau)$ on the right side of (8.7) with a simpler expansion in which the error terms are linear in $\nu(s) - \nu(0)$, $\tau(s) - \tau(0)$, s. Conditions (8.17) now cause a very satisfying cancellation of terms on the right side of (8.7) (which will render the s^{-1} innocuous); the resulting equation has the form

$$\hat{N}_\nu(\nu(s), \tau(s), \alpha, s)s^{-2}(s^3\tau')' = s^{-1}Q(s) - \hat{N}_s(\nu(s), \tau(s), \alpha, s) \tag{8.21}$$

where $Q(s)$ is a quadratic form in

$$\tau(s) - \tau(0), \qquad s\tau'(s), s \tag{8.22}$$

with coefficients of the form $\overline{N}_{\nu\nu}(s)$, etc. Indeed, if the material is everywhere isotropic, then

$$Q(s) = \frac{1}{2}(\overline{N}^{\nu\nu} - 2\overline{N}^{\nu\tau} + \overline{N}^{\tau\tau})(s\tau')^2. \tag{8.23}$$

Now the Ascoli–Arzelà Theorem implies that the mappings from u to

$$s^\kappa\tau' \;\forall\, \kappa > 0, \tau, \quad s^{-\lambda}[\tau - \tau(0)] \;\forall\, \lambda < 1, \tag{8.24}$$

induced by (8.12)–(8.14), are completely continuous from C^0 to C^0. Since expressions such as $s^{-1}[\tau - \tau(0)]^2$, which appear in $s^{-1}Q$, can be written as products as of terms from (8.24) and since $\hat{N}$ and $\hat{T}$ are twice continuously differentiable, it follows that the operator $f(l, \alpha, \cdot)$ appearing in (8.15) is also completely continuous. This result is central for several useful theorems, which we now discuss.

For clarity, let us restrict our attention to plates for which the nonuniformity arises in a particularly simple and natural way through the constitutive equations

$$\hat{N}(\nu, \tau, \theta, s) = h(s)N^{\#}(\nu, \tau, \theta), \qquad \hat{T}(\nu, \tau, \theta, s) = h(s)T^{\#}(\nu, \tau, \theta). \tag{8.25}$$

$h(s)$, which may be regarded as the thickness of the plate at s, is assumed to be positive and continuously differentiable on [0, 1]. Then (8.21) reduces to

$$N^{\#}_{\nu}(\nu(s), \tau(s), \alpha)s^{-2}(s^3\tau')' = s^{-1}Q^{\#}(s) - [h'(s)/h(s)]N^{\#}(\nu(s), \tau(s), \alpha) \tag{8.26}$$

where $Q^{\#}$ is obtained from Q by replacing $\hat{N}$, $\hat{T}$ by $N^{\#}$, $T^{\#}$. We regard (8.26) and its corresponding operator form (8.15) as parametrized by l, α, and the function h. Thus the problem has an infinite-dimensional parameter.

The linearization of the version of (8.15) associated with (8.26) about the solution $u = 0$ (corresponding to $\tau = l$) when l, α, h are fixed positive constants $\bar{l}$, $\bar{\alpha}$, $\bar{h}$ is

$$N^{\#}_{\nu}(l, l, \alpha)v = 0. \tag{8.27}$$

Let h_k represent any k-parameter approximation to h in C^1.

That (8.27) has the unique solution $v = 0$ and that $(l, \alpha, h_k, u) \to f(l, \alpha, h_k, u)$ is completely continuous from $(0, \infty) \times (0, \infty) \times \mathbb{R}^k \times C^0$ to C^0 supports the following connectivity theorem (see [2], Theorem 3.2): *Equation* (8.15) *associated with* (8.26) *has a connected family* C *of solution pairs* ((l, α, h), u) *properly containing* $((\bar{l}, \bar{\alpha}, \bar{h}), 0)$ *each point of which has infinite dimension. In a neighborhood of* $((\bar{l}, \bar{\alpha}, \bar{h}), 0)$, C *is an infinite-dimensional surface. The restriction of* C *obtained by confining the parameters* (l, α, h) *to a line is a connected set having at least one of the following three properties:* (i) *It is unbounded,* (ii) *It approaches the set where* τ *or* ν (*in terms of* u) *vanish,* (iii) *It is a closed figure in the sense that it can be mapped onto a circle.* (C enjoys further properties that we do not pause to spell out [2].)

This theorem (in contrast to that stated at the beginning of this section) does not say that there is a solution for each (l, α, h). It may happen that u can become unbounded while the (l, α, h) stay bounded on C or that alternative (iii) holds. These possibilities are not incompatible with general existence results quoted in the paragraph containing (8.10) because those results say nothing about connectivity. Only when (8.10) holds can we combine the general existence and uniqueness theorem with the connectivity theorem to conclude that all solution pairs lie on a connected set C with exactly one u corresponding to each (l, α, h).

To see why it is hard to obtain this conclusion when (8.10) does not hold we consider a special class of isotropic (hyperelastic) materials of the form (8.25) for which

$$\begin{aligned} N^{\#}(\nu, \tau, \theta) &= A\nu^{-a}\tau^{-a+1} - B\nu^{-b} + C\nu^{c} + D\nu^{d}\tau^{d+1}, \\ T^{\#}(\nu, \tau, \theta) &= N^{\#}(\tau, \nu, \Theta), \end{aligned} \tag{8.28}$$

where A, B, C, D, a, b, c, d are given positive-valued functions of θ. It follows that $N^{\#}_{\nu} > 0$, but that (8.10) need not hold. ((8.10) does not hold for D sufficiently large when ν and τ are sufficiently large.)

Now (8.28) implies that

$$-\left(\frac{1}{a} + \frac{1}{b}\right)\nu \leq N^{\#}(\nu, \tau, \theta)/N^{\#}_{\nu}(\nu, \tau, \theta) \leq \left(\frac{1}{c} + \frac{1}{d}\right)\nu. \tag{8.29}$$

Thus (8.15) associated with (8.26) and (8.29) has the form (see (8.11)–(8.14), (8.23))

$$u(s) = \{F[u](s)\}\left[s^{-5/2}\int_0^s t^2u(t)\,dt\right]^2 - [h'(s)/h(s)]G[u](s) \tag{8.30}$$

where

$$|G[u](s)| \leq \text{const}\left\{l + \frac{1}{2}\int_0^s (s^{-2}+1)t^2u(t)\,dt - \frac{1}{2}\int_s^1 (1-t^2)u(t)\,dt\right\}. \tag{8.31}$$

Here F is well-behaved for u reasonably near 0. But the form of the right side of (8.30) does not lend itself to a proof that u is bounded for all values of the parameters. The culprit is the term l in the right side of (8.31). It is absent only when $h' = 0$. The issues involved are made clear by the study of the multiplicity of solutions of primitive one-dimensional analogs of (8.30), such as

$$v = \pm v^2 + \lambda(v + 1). \tag{8.32}$$

Here v corresponds to u and λ to h'/h. The multiple solutions of (8.32) can easily be found explicitly.

That (8.30) might have solutions like those of (8.32) when (8.10) does not hold suggests that nonuniformity of the plate can induce new kinds of instabilities, the presence of which illuminates the role of constitutive restrictions such as the strong ellipticity condition and condition (8.10). This question will be pursued elsewhere in a numerical study.

An alternative formulation in place of (8.15) is obtained by setting $r = \mathbf{r}\cdot\mathbf{i}$. Then for trivial states $\nu = r'$, $\tau = r/s$ and

$$s^{-2}(s^3\tau'(s))' = s^{-1}[(sr')' - s^{-1}r]. \tag{8.33}$$

The operator in the brackets on the right side of (8.33) is that for the Bessel function J_1. The integral equation for $w = s^{-1/2}[(sr')' - s^{-1}r]$ is treated in [3].

In our treatment of (8.15) we have sought continuous u's. Such u's correspond to r's for which $r(0) = 0$. We have thereby excluded from consideration those solutions representing cavitation, the process in which a hole opens at $s = 0$. See [8].

The basic approach of this section can also handle beams.

From the viewpoint of global bifurcation theory the most important feature of this section is that a much more complicated version of the analysis carried out here shows that the full set of governing equations for a plate isotropic at its center can be cast as integral equations of the form (8.15):

$$\mathbf{u} = \mathbf{f}((l, \alpha, h), \mathbf{u}) \tag{8.34}$$

with $\mathbf{f}((l, \alpha, h), \cdot)$ completely continuous from $[C^0]^3$ to itself for each (l, α, h) and with the restriction of f to $K \times [C^0]^3$, where K is a closed and bounded subset of $(0, \infty) \times (0, \infty) \times \mathbb{R}^n$, also completely continuous. The details follow the lines of [3, 10]. This result enables us to invoke the theory described in Section 1.

9. THE LINEARIZATIONS

The linearizations of (3.7)–(3.9), (5.1) about the trivial solutions discussed in Sections 7 and 8 are the following systems for the "variations" (ψ_1, η_1):

$$[M^0_\eta\eta_1 + M^0_\mu\psi_1']' - N_1[0, 0, \theta_0; l, \beta](\nu_0\psi_1 + \eta_1) = 0 \tag{9.1b}$$

$$\{s[M^0_\eta\eta_1 + M^0_\mu\psi_1' + M^0_\sigma\psi_1/s]\}' - [\Sigma^0_\eta\eta_1 + \Sigma^0_\mu\psi_1' + \Sigma^0_\sigma\psi_1/s] - sN^0(\nu^0\psi_1 + \eta_1) = 0, \tag{9.1p}$$

$$H^0_\eta\eta_1 + H^0_\mu\psi_1' = -N_1[0, 0, \theta_0; l, \beta]\psi_1, \tag{9.2b}$$

$$H^0_\eta\eta_1 + H^0_\mu\psi_1' + H^0_\sigma\psi_1/s = -N^0\psi_1. \tag{9.2p}$$

Here, e.g. $M_\eta^0(s) \equiv M_\eta((s\tau_0(s))', 0, 0, \tau_0(s), 0, 0, \alpha, s)$ for plates, where τ_0 is the solution whose existence was established in Section 8.

The substitution of (9.2) into (9.1) yields a Sturm–Liouville equations for ψ_1 parametrized by the boundary values for temperature and by constitutive functions (which account for thickness variations). ψ_1 must satisfy the boundary conditions

$$\psi_1(0) = 0 = \psi_1(1). \tag{9.3}$$

If the plate is everywhere isotropic and uniform, then (5.11p) reduces (9.1p), (9.2p) to

$$(M_\mu^0 H_\eta^0 - M_\eta^0 H_\mu^0)[(s\psi_1')' - \psi_1/s] + M_\eta^0(H_\mu^0 - H_\sigma^0)(\psi_1' - \psi_1/s) + N^0(H_\mu^0 - M_\eta^0)s\psi_1' + N^0 H_\sigma^0 \psi_1 + N^0(N^0 - lH_\eta^0)s\psi_1 = 0. \tag{9.4p}$$

If $H_\mu^0 = H_\sigma^0 = M_\eta^0 = 0$, then (9.4p) reduces to a Bessel equation having nontrivial solutions

$$J_1\left(\sqrt{\frac{N^0(N^0 - lH_\eta^0)}{M_\mu^0 H_\eta^0}}\, s\right) \tag{9.5p}$$

whenever the radical in (9.5p) is a zero of the Bessel function J_1. The values of α for which (9.1p), (9.2p) has nontrivial solutions are the eigenvalues of the linearized problem. The number and disposition of these eigenvalues depend upon the way the constitutive functions depend on θ. It is quite reasonable to require that $\hat{N}$ be decreasing in θ, but there is no doctrine governing how the stiffnesses $\hat{M}_\mu$ and $\hat{H}_\eta$ should depend on θ. Thus even for the simplest case, leading to (9.5p), the eigenvalues can have the most varied of distributions. In particular, there could be but a finite number of eigenvalues and there could be several eigenvalues corresponding to the same eigenfunction. We illustrate these possibilities in Fig. 9.6p. Analogous results hold for beams.

The eigenvalues shown in Fig. 9.6p are simple because the curves A and B cross the horizontal lines at ordinates $j_0, j_1, \ldots$ transversally. It is eminently possible for a given function R^0 to have any given degree of contact with a horizontal line at level j_k. In this case the corresponding eigenvalue has algebraic multiplicity equal to the (possibly infinite) degree of contact. But in a way that could be stated with mathematical precision, the curves R^0 typically cross the horizontal lines through $j_0, j_1, \ldots$ transversally. Our best qualitative results will only apply to connected sets of nontrivial solution pairs bifurcating from simple eigenvalues.

Note that Fig. 9.6p only shows the restriction of R^0 to the line l = const. R^0 actually defines a surface over the (l, α)-plane; its intersections with this plane are *eigencurves*. (Strictly speaking, these collections of eigenvalues (l, α) need not lie on curves, in general; in many cases, however, one can prove that they do so.)

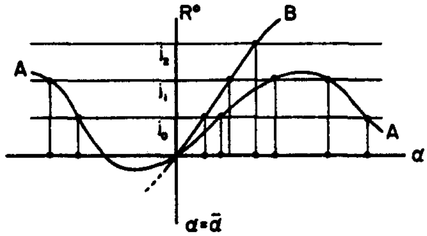

Fig. 9.6p. Eigenvalues for problems leading to (9.5p) with l fixed. R^0 stands for the radical in (9.5p). $j_0, j_1, j_2, \ldots$ are the zeros of J_1. $\bar{\alpha}$ is a temperature at which the unstressed plate has radius l. A and B represent typical curves giving R^0 as a function of θ_1. The eigenvalues are the abscissae at which $R^0 = j_k, k = 0, 1, 2, \ldots$. It is reasonable to assume that N^0 is decreasing with θ. Our choice of $\bar{\alpha}$ implies that N^0 and hence R^0 vanish when $\alpha = \bar{\alpha}$. Eigenvalues with $\alpha < \bar{\alpha}$ correspond to shear instabilities (see [7]).

In the corresponding problem for beams, we have at least three basic parameters l, α, β, to which we may add the constitutive functions. In this case we obtain eigensurfaces of various dimensions. An eigenvalue lying in an eigensurface is *simple* if it is a simple eigenvalue for the one-parameter problem obtained by restricting the parameters to any straight line transversal to the eigensurface at the eigenvalue in question.

10. NODAL PROPERTIES

The results obtained in Sections 6 and 8 show that the integral equations for beams and for circular plates *isotropic at the center* are nice enough for us to apply the global bifurcation theory outlined in Section 1. Thus, bifurcating from each eigenvalue of odd multiplicity of the linearization is a "maximally" connected family of nontrivial solutions pairs (corresponding to buckled states) possessing the global properties described in Section 1. ("Most" eigenvalues are not only odd, but are actually simple. See Section 9.) But this result is not particularly useful or exciting by itself. We want to know what solutions look like on each such connected set.

We accordingly examine the specific behavior of ψ on each connected set of solution pairs by studying its nodal properties. If the only zeros of ψ at one place on a connected set of solution pairs are simple, then for ψ to suffer a change in the number of zeros as the solution pair moves over a connected set of solution pairs, it must pass through a place where ψ has a double zero, i.e. there must be a ζ in $[0, 1]$ where

$$\psi(\zeta) = 0 = \psi'(\zeta). \tag{10.1}$$

Now consider the beam equations in the following form

$$\psi' = \hat{\mu}(N(1) \cos \psi, -N(1) \sin \psi, M, \Gamma, \theta, s), \tag{10.2b}$$

$$M' = N(1)[\hat{\nu} \sin \psi + \hat{\eta} \cos \psi] \tag{10.3b}$$

where the arguments of $\hat{\nu}$ and $\hat{\eta}$ in (10.3b) are those of $\hat{\mu}$ in (10.2b). (See (3.6b), (3.7b), (3.9b), (5.7b).) Now (5.2), (5.10b), (10.2b), and (10.3b) imply that (10.1) holds if and only if

$$\psi(\zeta) = 0 = M(\zeta). \tag{10.4b}$$

Thus (10.2b)–(10.4b) is an initial value problem for (ψ, M), necessarily having a unique solution, which we identify as $(\psi, M) = (0, 0)$. A technically more delicate analysis, needed to accommodate the singularity at $s = 0$, yields the same conclusion for plates. (The procedure is essentially that given in ([3], Sec. 7). In this analysis the isotropy at the center again plays a critical role.)

Thus we find that ψ can change its nodal properties only at the family of trivial solutions. Now in Section 1 we noted that near a bifurcation point solutions ψ "looks like" the eigenfunction of the linearized problem provided the eigenvalue is simple. Thus if a problem has a bifurcation point with a simple eigenvalue, then ψ inherits the nodal structure of the corresponding eigenfunction and preserves it everywhere on the bifurcating connected family of solution pairs. Moreover this connected family enjoys the further properties described in Section 1. In particular, if we restrict our attention to plate problems parametrized solely by α, then we know that the connected family of solution pairs bifurcating from a simple eigenvalue α^* must be unbounded in the space $[0, \infty) \times C^0 \times C^0$ of $(\alpha, (\psi, M))$ if the linearized problem has no other eigenvalues with eigenfunctions equal to that for α^*. (This result is a consequence of a generalization of the fact that near a bifurcation point with a simple eigenvalue the solution looks like the eigenfunction.)

11. LOCAL POSTBUCKLING BEHAVIOR

We now study the behavior of solutions of the beam equations in a neighborhood of a bifurcation point. To obtain equations with constant coefficients, we limit our attention to uniform beams with $\beta = 0$. Our basic results can be extended to nonuniform beams with $\beta \neq 0$ and to nonuniform plates. When $\beta = 0$ the trivial solution (see Section 7) is defined by

$$\psi = 0, \quad \nu_0 = l, \quad \eta_0 = 0, \quad \theta_0 = \alpha_0. \tag{11.1}$$

We just study the case in which l is held fixed.

For simplicity we strengthen (5.10b) by assuming that

$$\begin{aligned} &\hat{N} \text{ is even in } \eta, \mu, \gamma;\\ &\hat{H} \text{ is odd in } \eta \text{ and even in } \mu, \gamma;\\ &\hat{M} \text{ is odd in } \mu \text{ and even in } \eta, \gamma;\\ &\hat{\Gamma} \text{ is odd in } \gamma \text{ and even in } \eta, \mu. \end{aligned} \tag{11.2}$$

Next we introduce a small parameter

$$\epsilon \equiv \langle \psi, \sqrt{2} \sin n\pi s \rangle \equiv \sqrt{2} \int_0^1 \psi(s) \sin n\pi s \, ds, \tag{11.3}$$

which measures the amplitude of ψ near a bifurcation point $(\alpha_0, 0)$ where α_0 is an eigenvalue of the linearization corresponding to eigenfunction $\sqrt{2} \sin n\pi s$. (We suppress the dependence of ϵ on n.) We seek solution pairs of the beam equations satisfying the side condition (11.3) in the form

$$\begin{aligned} &\alpha(\epsilon) = \alpha_0 + \epsilon\alpha_1 + (\epsilon^2/2!)\alpha_2 + \dots, \quad \psi(s, \epsilon) = \epsilon\psi_1(s) + (\epsilon^2/2!)\psi_2(s) + \dots,\\ &\nu(s, \epsilon) = l + \epsilon\nu_1(s) + (\epsilon^2/2!)\nu_2 + \dots, \quad \eta(s, \epsilon) = \epsilon\eta_1(s) + (\epsilon^2/2!)\eta_2(s) + \dots. \end{aligned} \tag{11.4}$$

(See [11].) We substitute (11.3) into (11.2) and into the governing equations in the original form listed at the beginning of Section 6, differentiate the resulting equations once with respect to ϵ and set $\epsilon = 0$. Using (11.2) we obtain a problem equivalent to a duly specialized version of that of Section 9:

$$\langle \psi_1, \sqrt{2} \sin n\pi s \rangle = 1, \tag{11.5}$$

$$N^0_\nu \nu_1 + N^0_\theta \theta_1 = N^0_\nu \nu_1(1) + N^0_\theta \theta_1(1) \quad \text{(from (3.6b) and)5.1b))}, \tag{11.6}$$

$$H^0_\eta \eta_1 + N^0 \psi_1 = 0 \quad \text{(from (3.7b) and (5.1b))}, \tag{11.7}$$

$$M^0_\mu \psi_1'' - N^0(l\psi_1 + \eta_1) = 0 \quad \text{(from (3.9b) and (5.1b))}, \tag{11.8}$$

$$\Gamma^0_\gamma \theta_1' = \Gamma^0_\gamma \theta_1'(1), \tag{11.9}$$

$$\psi_1(0) = 0 = \psi_1(1) \quad \text{(from (2.10))}, \tag{11.10}$$

$$\theta_1(0) = \alpha_1 = \theta_1(1) \quad \text{(from (4.4b) with } \beta = 0\text{)}, \tag{11.11}$$

$$\int_0^1 \nu_1(s) \, ds = 0 \quad \text{(from (2.9) and (2.3))}. \tag{11.12}$$

This system has the nontrivial solution

$$\nu_1 = 0, \quad \eta_1 = -(N^0/H^0_\eta)\psi_1, \quad \psi_1 = \sqrt{2} \sin n\pi s, \quad \theta_1 = \alpha_1, \tag{11.13}$$

when α_0 satisfies

$$\frac{N^0}{M^0_\mu H^0_\eta}(N^0 - lH^0_\eta) = n^2\pi^2. \tag{11.14}$$

Differentiating the governing equations twice with respect to ϵ and then setting ϵ equal to 0 and using (11.13) we obtain

$$\langle \psi_2, \sqrt{2} \sin n\pi s\rangle = 0, \tag{11.15}$$

$$N_\nu^0 \nu_2 + N_\theta^0 \theta_2 = -N_{\eta\eta}^0 (N^0/H_\eta^0)^2 \psi_1^2 - N_{\mu\mu}^0 (\psi_1')^2 - N_{\theta\theta}^0 \alpha_1^2 + N_\nu^0 \nu_2(1) + N_\theta^0 \theta_2(1) - N^0 \psi_1^2, \tag{11.16}$$

$$H_\eta^0 \eta_2 + N^0 \psi_2 = 2 H_{\eta\mu}^0 (N^0/H_\eta^0) \psi_1 \psi_1' + 2H_{\eta\theta}(N^0/H_\eta^0)\alpha_1 \psi_1 - 2N_\theta^0 \alpha_1 \psi_1. \tag{11.17}$$

$$M_\mu^0 \psi_2'' - N^0(l\psi_2 + \eta_2) = 2M_{\eta\mu}^0 (N^0/H_\eta^0)(\psi_1\psi_1')' - 2M_{\mu\theta}^0 \alpha_1 \psi_1'' + 2N_\theta^0 \alpha_1 [l - (N^0/H_\eta^0)]\psi_1, \tag{11.18}$$

$$\Gamma_\gamma^0 \theta_2' = \Gamma_\gamma^0 \theta_2'(1), \tag{11.19}$$

$$\psi_2(0) = 0 = \psi_2(1), \tag{11.20}$$

$$\theta_2(0) = \alpha_2 = \theta_2(1), \tag{11.21}$$

$$\int_0^1 \nu_2(s)\, ds = [l - 2(N^0/H_\eta^0)] \int_0^1 \psi_1(s)^2\, ds = l - 2(N^0/H_\eta^0). \tag{11.22}$$

We now solve (11.17) for η_2, which we substitute into (11.18), multiply this version of (11.18) by ψ_1, and then integrate the resulting equation by parts over [0, 1]. We get

$$\alpha_1 \left\{ \left(\frac{N^0}{H_\eta^0}\right)^2 H_{\eta\theta}^0 - M_{\mu\theta}^0 (n\pi)^2 - N_\theta^0 \left\{ l - \left(\frac{N^0}{H_\eta^0}\right)\right]\right\} = 0. \tag{11.23}$$

Let us assume that the term in braces does not vanish. (In a mathematically precise sense it has zero probability of vanishing.) Then $\alpha_1 = 0$. We can now solve the resulting simplified version of (11.17) and (11.18) for ψ_2 and η_2. Note that (11.15) ensures that the resulting expressions do not contain terms proportional to ψ_1. We also find that $\theta_2 = \alpha_2$ and obtain an explicit expression for ν_2.

Differentiating the governing equations thrice with respect to ϵ and then setting ϵ equal to 0, and using the solutions already obtained we obtain (among other equations)

$$\begin{aligned} H_\eta^0 \eta_3 + N^0 \psi_3 = {} & -3H_{\nu\eta}^0 \nu_2 \eta_1 - 3H_{\eta\mu}^0 (\psi_2' \eta_1 + \psi_1' \eta_2) \\ & - 3H_{\eta\theta}^0 \alpha_2 \eta_1 - H_{\eta\eta\eta}^0 \eta_1^3 \\ & - 3H_{\eta\mu\mu}^0 \eta_1 (\psi_1')^2 + 3[N_{\eta\eta}^0 (N^0/H_\eta^0)^2 \psi_1^2 + N_{\mu\mu}^0 (n\pi)^2 \\ & - N_\nu^0 \nu_2(1) - N_\theta^0 \alpha_2]\psi_1 + 3N^0 \psi_1^3, \end{aligned} \tag{11.24}$$

$$\begin{aligned} M_\mu^0 \psi_3'' - N^0(l\psi_3 + \eta_3) = {} & -3M_{\nu\mu}^0 (\nu_2 \psi_1')' - 3M_{\eta\mu}^0 (\eta_2 \psi_1' + \eta_1 \psi_2')' \\ & + M_{\mu\theta}^0 \alpha_2 \psi_1'' - 3M_{\eta\eta\mu}^0 (\eta_1^2 \psi_1')' - M_{\mu\mu\mu}^0 (\psi_1')^3 \\ & + 3(l\psi_1 + \eta_1)[-N_{\eta\eta}^0 (N^0/H_\eta^0)^2 \psi_1^2 - N_{\mu\mu}^0 (\psi_1')^2 \\ & + N_\nu^0 \nu_2(1) + N_\theta^0 \alpha_2 - N^0 \psi_1^2], \end{aligned} \tag{11.25}$$

We solve (11.24) for η_3, which we substitute into (11.25), multiply this equation by ψ_1, and integrate the resulting equation by parts over [0, 1]. We get a linear equation for α_2, whose coefficient is the nonzero term in braces in (11.23) (as a comparison of (11.17), (11.18) with (11.24), (11.25) shows). Therefore we can find α_2 in terms of all the constitutive variables appearing in (11.23)–(11.25). (Note that we do not have to solve (11.25) to find α_2.)

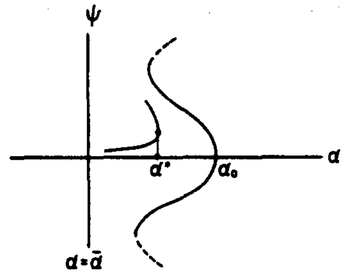

Fig. 11.26. The bifurcation from α_0 is subcritical.

The sign of α_2, in particular, depends on all these constitutive variables. For most of these variables we have no information, experimental, intuitive, or otherwise, on their behavior or even their signs. When α_2 is positive the bifurcation is supercritical and when α_2 is negative the bifurcation is subcritical as shown in Fig. 11.26.

In this case an imperfection analysis would indicate the possibility of an instability occurring at a temperature $\alpha*$ below the first buckling temperature α_0. Of course such phenomena cannot be found in theories relying on linear constitutive equations, in which case a straightforward analysis of (11.24), (11.25) when we require the vanishing of all derivatives of constitutive functions of order two and higher leads to

$$N^0_\theta \alpha_2 = N^0/2 - N^0_\nu(l - 2N^0/H^0_\eta). \qquad (11.27)$$

If $N^0_\theta < 0$, then $\alpha_2 > 0$ and there is only supercritical bifurcation.

12. COMMENTS

Under mild constitutive assumptions we could give a quite specific description of the location of the connected families of solution pairs in solution-parameter space. In particular, for problems parametrized by a single temperature such assumptions would imply that the families ultimately "move" to the right as illustrated in Fig. 12.1. Consequently unbounded branches have solution pairs for each α sufficiently large. The methods for proving these facts are given by ([17], Section 8).

There is very little experimental information on the nonlinear constitutive response of real materials under combined states of strain. This fact caused us no inconvenience because we studied a whole class of constitutive relations at one time. The virtue of this generality is that it exhibits thresholds in material response corresponding to qualitatively different behavior of solutions. For example, one such threshold distinguishes between sub- and supercritical bifurcation in Section 11. We note that the possibility of thresholds does not arise if we restrict our attention solely to materials with linear stress–strain laws as is done in such classical treatments as that of [15]. The availability of such thresholds can be used to guide experimental programs. Instead of seeking a specific constitutive function for a given material, one can classify materials by their susceptibility to different sorts of instabilities.

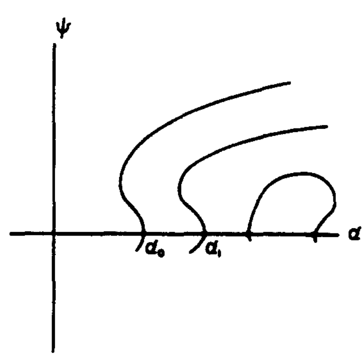

Fig. 12.1.

In traditional treatments of thermal buckling of plates the effect of temperature is essentially equivalent to that of a pressure applied to the edge. It is clear from a comparison of our work with [3] that such an analogy is not valid for exact nonlinear problems because the parameters of temperature and pressure would appear in the governing equations in different places. In particular, the curves corresponding to A and B of Fig. 9.6p for the buckling under pressure would have a different form even though the material response is the same. There would in general be no connection at all between the bifurcation diagrams of the form of Fig. 12.1 for pressure loadings and temperature loadings.

There are many nonlinear models for beams and even more for plates (see [9, 12], e.g.). Most are obtained by discarding certain terms in strain-displacement relations and in expressions for angles that are deemed negligible. Since we are studying very large deformations we cannot avail ourselves of this luxury. This is no disadvantage: Were we to choose any particular model, then we might miss interesting effects appearing only in other models. Moreover, we would immerse ourselves in the ongoing controversy of the relative merits of different models. We avoid these pitfalls by defining the configuration a beam as a curve equipped with cross-sections (not necessarily normal to the curve) and the configuration of a plate as a surface equipped with vectors (characterizing the orientations of vectors originally normal to the surface). Then the kinematics of deformation proceeds inexorably and exactly from these definitions, which encompass most of the special approximate theories studied. When this kinematics is combined with correct equations of equilibrium and general invariant constitutive relations, our problems are posed in settings both more general and more precise than those used in the many special theories. This generality has the virtue that it does not obscure the structure of the equations, which reflects the basic concepts of mechanics: strains, forces, couples, and material response.

Of course, the qualitative results obtained in the main body of this paper and the quantitative results of the sort obtained in Section 11 and of the sort alluded to in the opening paragraph of this section do not exhaust the subject. It would be important to find out where the bifurcating branches of Fig. 12.1 go and in particular, to get a sharp estimate of the smallest α in Fig. 12.1 corresponding to a buckled state. In general, to obtain such information one must resort to numerical computation based on the choice of particular constitutive equations. (The availability of detailed nodal properties can significantly aid such computations.) How does one obtain such constitutive equations? We suppose that we know the three-dimensional constitutive equations for a given material. These may be based on experimental results for a real material (which are very hard to come by) or may represent some concrete idealization. (Incidentally, a certain amount of thermoplastic behavior can be accounted for in such constitutive equations as long as there is monotonicity in the loading.) Various beam and plate models can be obtained by constraining the cross-sectional behavior of the position and temperature fields (e.g. by assuming that the position field is linear and the temperature field is constant in the transverse coordinates) and then by integrating the constrained versions of the constitutive equations over the section to get the reduced constitutive equations for beams and plates. This general process is described in detail in [3, 10]. In particular, a typical family of ideal constitutive equations for plates that can be generated in this way are the following:

$$\hat{N} = \int_{-h/2}^{h/2} (\nu - y\psi')f\,dy, \qquad \hat{T} = \int_{-h/2}^{h/2} (\tau - y\sigma)f\,dy, \tag{12.2a,b}$$

$$\hat{M} = -\int_{-h/2}^{h/2} y(\nu - y\psi')f\,dy, \qquad \hat{\Sigma} = -\int_{-h/2}^{h/2} y(\tau - y\sigma)\,dy, \tag{12.2c,d}$$

$$\hat{H} = \eta \int_{-h/2}^{h/2} [BI^b + C(I^2 - 2II)^c(I - 1) - E]\,dy, \tag{12.2e}$$

where

$$f \equiv AIII^{-a} + BI^{b} + C(I^2 - 2II)^{c}(I - 1) + DIII^{d} - E, \tag{12.2f}$$

I, II, III are the principal invariants of the Green deformation tensor with components

$$\begin{pmatrix} (\nu - y\psi')^2 + \eta^2 & \eta & 0 \\ \eta & 1 & 0 \\ 0 & 0 & (\tau - y\sigma)^2 \end{pmatrix}, \tag{12.2g}$$

A, B, C, D are positive functions depending on θ and γ, $E = A + 3^bB + 3^cC + D$, and a, b, c, d are positive numbers. Analogous equations for beams are obtained by dropping (12.2b,d) and replacing the entry in the (3,3)-slot of (12.2g) with 1. (Specializations of these equations yield (8.28).) We may supplement these equations with one for $\hat{\Gamma}$, say the Fourier heat conduction law, for which $\hat{\Gamma}$ is linear in γ. For certain values of the exponents a, b, c, d the integrals in (12.2) can be evaluated explicitly. When this is not possible, it is an easy matter to expand these integrals in powers of h and discard terms of sufficiently high order in h. In any case, specific choices of the exponents and moduli in (12.2) lead to specific constitutive equations and to specific formulas for the curves in Fig. 9.6. Right now an extensive numerical computation of the global form of bifurcating branches for buckling problems like ours with constitutive equations like (12.2) is being carried out by I. Babuška and W. Rheinboldt. The procedure is quite delicate.

We did not employ constitutive restrictions coming from a version of a Second Law of Thermodynamics because we had no need for them. Instead, we employed an alternative set of conditions associated with the strong ellipticity condition. These conditions may be regarded as characterizing a certain class of material stabilities.

We could generalize our problems in many ways: by studying other kinds of boundary conditions, by allowing the beam to deform in space (see [6]), and by introducing another temperature variable to characterize the variation of temperature across a section. Only in the latter case do we fail to get a detailed qualitative picture of global behavior because we lack a suitable generalization of Section 10. This problem has been formulated and analyzed as far as possible in [10].

The lowest buckling mode usually supplies the most important information about stability. What physical significance do the higher modes have? Thought likely to be unstable, they may be regarded as organizing the dynamics of the corresponding motion in much the same way that equilibrium points, stable and unstable, organize the phase portrait of an autonomous differential equation for a nonlinear oscillator.

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